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Analysis of nonlinear duopoly game with heterogeneous players

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Abstract

The paper considers a nonlinear duopoly game with heterogeneous players, boundely rational and naive expectations. A duopoly game is modelled by two nonlinear difference equations. The existence and stability of the equilibria of this system are studied. The complex dynamics, bifurcations and chaos are displayed by computing numerically the largest Lyapunov exponents, sensitive dependence on initial conditions and fractal dimension of the chaotic attractor.

Keywords: Discrete dynamical system; Cournot duopoly games; Heterogenous expectations; Complex dynamics

1. Introduction

Monopoly market has such a market structure in which a trade is completely controlled by several firms. The fewness firms manufacture the same or homogeneous products and they must consider not only the demand of market, but also the actions of the competitors. In 1838, the French Mathematician Cournot was first to introduce the Cournot model which was most widely used mathematical representations of duopoly market. Cournot (1838) investigated the case that each player(firm) was provided with naive expectations in duopoly. He assumed that each firm is able to produce precisely the quantity of production by its rival's output. But it is impossible that all players are naive. There, different players' expectations are proposed: naive player, bounded rational player and adaptive (Agiza and Elsadany, 2004a). So each player adopts his expectations to adjust his outputs in order to maximize his profit.

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Recently, several works of the Cournot Model have been done. Many conclusions have shown that Cournot Model has very abundant dynamical behavior such as cyclic, bifurcation and chaos under the different expectations. So expectations play an important role in studying this economic phenomena.

Several cases of each player with homogeneous expectations have been considered by some authors (see puu, 1991; Puu, 1998; Agiza, 1998, 1999; Agiza et al., 2001, 2002; Kopel, 1996). Puu (1991, 1998) firstly found a variety of complex dynamics arising in the Cournot duopoly case including the appearance of attractors with fractal dimension. In Agiza (1998, 1999) and Agiza et al. (2001, 2002), M. Kopel studied the dynamics of oligopoly models with more players and other modification of Puu's model. The game models with heterogeneous players also have been studied (Agiza and Elsadany, 2003, 2004b; Leonard and Nishimura, 1999; Den-Haan, 2001).

Agiza and Elsadany (2003) have applied the technique of Onazaki et al. (in press) to study the dynamics of Cournot duopoly model which contains two heterogenous players, one boundedly rational and the other naive. By modifying the linear cost function, we study the case of this model with nonlinear cost function.

The paper is organized as follows. In Section 2we determine model the dynamical systems of a duopoly game with heterogeneous expectations, boundedly rational and native by a two-dimensional map. In Section 3we study the model of duopoly game. Explicit parametric conditions of the existence, local stability and bifurcation of equilibrium points will be given. In Section 4we show complex dynamic of this system via computing the largest Lyapunov exponents, sensitive dependence on initial conditions and fractal dimension of the chaotic attractor by numerical simulations.

2. Model

We consider that there are two firms producing goods which are perfect substitutes in a oligopoly market. Let $q_i(t)$, $i=1, 2$ represent the output of ith firm during period $t=0, 1, 2\ldots$ The duopolists determine the optimum quantities basing on their different expectations of the rival's output in the subsequent period. The retail price p, a linear inverse demand function, is determined from the total supply $Q(t)=q_1(t)+q_2(t)$ in period t

$$p = f(Q) = a - bQ \tag{1}$$

where $a>0$, $b>0$. The cost function has the nonlinear form

$$C_i(q_i) = c_i q_i^2, \qquad i = 1,2 \tag{2}$$

where c_i, $i=1, 2$ are positive shift parameters to the cost functions of the firm i, $i=$ 1, 2, respectively. Hence, the single profit of the ith firm in the single period is given by

$$\Pi_i(q_1, q_2) = q_i(a - bQ) - c_i q_i^2, \quad i = 1,2 \tag{3}$$

By (3), ith firm's output for period $t+1$ is decided by solving the optimization problem

$$q_1(t+1) = \arg\max \Pi_1(q_1(t), q_2^*(t+1)),$$

$$q_2(t+1) = \arg\max \Pi_2(q_1^*(t+1), q_2(t)) \tag{4}$$

where $q_i^*(t+1)$ represents the expectation of jth firm about jth firm's production during period $t+1$ $(i, j=1, 2, i\neq j)$.

Differentiating $\prod_i(q_i, q_j)$ with respect to q_i, we obtain the marginal profit of ith firm at the point (q_1, q_2) of the strategy space during period t

$$\Phi_i(t) = \frac{\partial \Pi_i(q_i, q_j)}{\partial q_i} = a - 2(b + c_i)q_i - bq_j, \qquad i,j = 1,2, \; i\neq j \tag{5}$$

This optimization problem has unique solution:

$$q_i = \frac{1}{2(b + c_i)}\left(a - bq_j\right) \tag{6}$$

For the player with boundedly rational, he determines quantities of production with the information of local profit maximizers and increases(decreases) its output if $\Phi_i(t)$ is positive (negative). In Dixit (1986), this adjustment mechanism has been called myopic by Dixit. The dynamic adjustment mechanism can be modeled as

$$q_i(t+1) = q_i(t) + \alpha_i q_i(t)\frac{\partial \Pi_i(q_i, q_j)}{\partial q_i}, \qquad t = 0, 1, 2, ..., \tag{7}$$

where α_i is a positive parameter and represents the speed of adjustment of ith firm. If the firm is a naive player, his expectation of production of rival will be the same as in previous period. Then the naive player decides his output according to Eq. (6).

In this paper, we consider each firm has different expectation to maximize his profit in duopoly game. We assume firm 1 is a boundedly rational player and firm 2 is a naive player. With above assumptions, we can express the process of duopoly game with heterogeneous players which is a two-dimensional nonlinear map $T(q_1, q_2)\rightarrow(q_1', q_2')$ defined as

$$T: \begin{cases} q_1' = q_1 + \alpha q_1 \dfrac{\partial \Pi_1(q_1, q_2)}{\partial q_1}, \\ q_2' = \dfrac{1}{2(b + c_2)}(a - bq_1) \end{cases} \tag{8}$$

where "$'$" denotes the unit-time advancement, that is if the right-hand side variables are productions of period t, then the left-hand ones represent productions of period $(t+1)$. Substituting Eq. (5) into discrete dynamic system (8), we have

$$\begin{cases} q_1' = q_1 + \alpha q_1(a - 2(b + c_1)q_1 - bq_2), \\ q_2' = \dfrac{1}{2(b + c_2)}(a - bq_1) \end{cases} \tag{9}$$

3. Analysis of model

In this paper, we are considering a economic model where only non-negative equilibrium points are meaningful. So that we only pay attention to the nonnegative fixed points of (9) i.e. the solution of the non-linear algebraic system as

$$\begin{cases} q_1(a - 2(b + c_1)q_1 - bq_2) = 0, \\ \dfrac{1}{2(b + c_2)}(a - bq_1) - q_2 = 0 \end{cases} \tag{10}$$

which is obtained by setting $q_i' = q_i$, $i=1, 2$ in system (9) We have two fixed points of system (10) $E_0 = \left(0, \frac{a}{2(b+c_2)}\right)$ and $E_1 = (q_1^*, q_2^*)$ where

$$q_1^* = \frac{a(b+2c_2)}{3b^2+4bc_1+4bc_2+4c_1c_2}, \quad q_2^* = \frac{a(b+2c_1)}{3b^2+4bc_1+4bc_2+4c_1c_2} \tag{11}$$

The fixed point E_0 is called boundary equilibria (Bischi and Naimzada, 1999). It is clear that the other fixed point E_1 is unique Nash equilibrium, located at the intersection of the two reaction curves which represent the locus of points of vanishing marginal profits in Eq. (5).

For studying the local stability of equilibrium point, we must consider the eigenvalues of the Jacobian matrix of the system (9) on the complex plane.

The Jacobian matrix of (9) at the point (q_1, q_2) has the form

$$J(q_1, q_2) = \begin{bmatrix} 1+\alpha[a-4(b+c_1)q_1-bq_2] & -\alpha b q_1 \\ -\frac{b}{2(b+c_2)} & 0 \end{bmatrix} \tag{12}$$

Theorem 1. *The boundary equilibria E_0 of system*(9) *is a unstable point.*

Proof. In order to prove this results, we consider the eigenvalues of Jacobian matrix J at E_0 which take the form.

$$J(E_0) = \begin{bmatrix} 1+\alpha\left[a-\frac{ab}{2(b+c_2)}\right] & 0 \\ -\frac{b}{2(b+c_2)} & 0 \end{bmatrix} \tag{13}$$

We have two eigenvalues of matrix $J(E_0)$, $\lambda_1 = 1+\alpha_{\left[a-\frac{ab}{2(b+c_2)}\right]}$ and $\lambda_2=0$. From the condition that a, b, c_i $(i=1, 2)$ are positive parameters, we have that $|\lambda_1|>1$. Then E_0 is unstable equilibrium point (saddle point) of system (9). This completes the proof the proposition. □

We now investigate the local stability of Nash equilibrium. The Jacobian matrix (9) at E_1 has the form

$$J(E_1) = \begin{bmatrix} 1+\alpha[a-4(b+c_1)q_1^*-bq_2^*] & -\alpha b q_1{*} \\ -\frac{b}{2(b+c_2)} & 0 \end{bmatrix} \tag{14}$$

Its characteristic equation is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(J)\lambda + \mathrm{Det}(J),$$

where Tr (J) is the trace and Det (J) is the determinant of the Jacobian matrix defined in (14),

$$\mathrm{Tr}(J) = 1+\alpha\left[a-4(b+c_1)q_1^*-bq_2^*\right] \text{ and } \mathrm{Det}(J) = -\frac{\alpha b q_1^*}{2(b+c_2)}$$

Since

$$\mathrm{Tr}^2(J)-4\mathrm{Det}(J) = \left[1+\alpha\left(a-4(b+c_1)q_1^*-bq_2^*\right)\right]^2 + \frac{2\alpha b q_1^*}{b+c_2} \tag{15}$$

It is clear that $\mathrm{Tr}^2(J)-4\mathrm{Det}(J)>0$, then the eigenvalues of Nash equilibrium are real.

If the eigenvalues of the Jacobian matrix of fixed point E_1 are inside the unit circle of the complex plane, Nash equilibrium E_1 is local stability. Using Jury's conditions (Puu, 2000), we have necessary and sufficient condition for local stability of Nash equilibrium which are the necessary and sufficient condition for $|\lambda_i|<1$, $i=1, 2$.

1.
$$1-\mathrm{Tr}(J)+\mathrm{Det}(J)=\frac{\alpha a(b+2c_2)}{2(b+c_2)}>0$$

2.
$$1+\mathrm{Tr}(J)+\mathrm{Det}(J)=2+\alpha a-4\alpha\left(b+c_1+\frac{b^2}{8(b+c_2)}\right)q_1^*-\alpha b q_2^*>0$$

3.
$$\mathrm{Det}(J)-1=-\frac{\alpha b q_1^*}{2(b+c_2)}-1<0$$

It is clear that the first condition and the third condition are always satisfied. Substituting (11) into the second condition, this condition becomes

$$\alpha<\frac{4(b+c_2)(3b^2+4bc_1+4bc_2+4c_1c_2)}{a(b+2c_2)(5b^2+4bc_1+4bc_2+4c_1c_2)} \tag{16}$$

By Eq. (16), we have Theorem 2 about local stability of Nash equilibrium point E_1.

Theorem 2. *The Nash equilibrium E_1 of system*(9) *is stable provided that*

$$\alpha<\frac{4(b+c_2)(3b^2+4bc_1+4bc_2+4c_1c_2)}{a(b+2c_2)(5b^2+4bc_1+4bc_2+4c_1c_2)}.$$

From Theorem 2, we can obtain the region of stability of the Nash equilibrium point E_1 about the model parameters. For example, an increase of the speed of adjustment of boundedly rational player with the other parameters held fixed has a destabilizing effect. In factor, an increase of α, staring from a set of parameters which ensures the local stability of the Nash equilibrium can bring out the region of the stability of Nash equilibrium point, crossing the flip bifurcation surface $\alpha=\frac{4(b+c_2)(3b^2+4bc_1+4bc_2+4c_1c_2)}{a(b+2c_2)(5b^2+4bc_1+4bc_2+4c_1c_2)}$. Obviously, the stability of Nash equilibrium point E_1 depends on the parameters of system. We also consider other case that the parameters α, b, c_1 and c_2 are fixed and parameter a which represents the maximum price of the production. In this paper, the case parameter α increases. Complex behaviors such as period doubling and chaotic attractors are generated where the maximum Lyapunov exponents of the system (9) become positive.

4. Numerical simulations

The main purpose of this section is to show the complicated dynamic features of the dynamics of a duopoly game (9) with heterogeneous players. The route that stability and period doubling bifurcation to chaos for system (9) is shown.

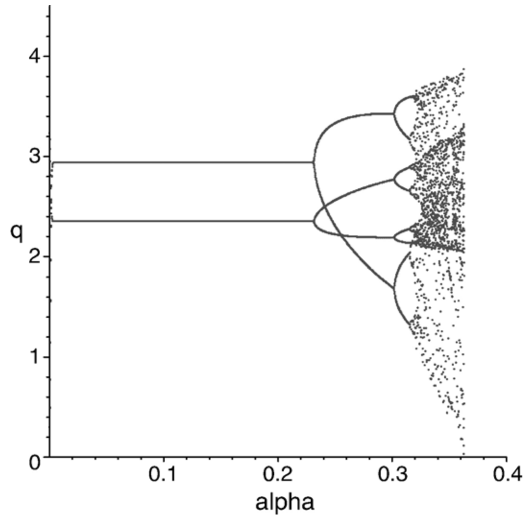

Fig. 1. The bifurcation diagram of the trajectories of the discrete dynamic system (9).

To provide some numerical evidence for chaotic behavior of system (9), we show several numerical results, such as its bifurcations diagrams, strange attractors, Laypunov exponents, sensitive dependence on initial conditions and fractal structure. In order to study the local stability properties of the equilibrium points conveniently, we take $a=10$, $b=1$, $c_1=0.3$ and $c_2=0.5$.

Fig. 1 shows the bifurcation diagram with respect to α (speed of adjustment of boundedly rational player), while other parameters are fixed. In Fig. 1, the Nash equilibrium $E_1=(2.94, 2.35)$ is locally stable for small values of the parameter α. If α increases, the Nash equilibrium E_1 becomes unstable and the bifurcation scenario is occurred. As α increases, infinitely many period-

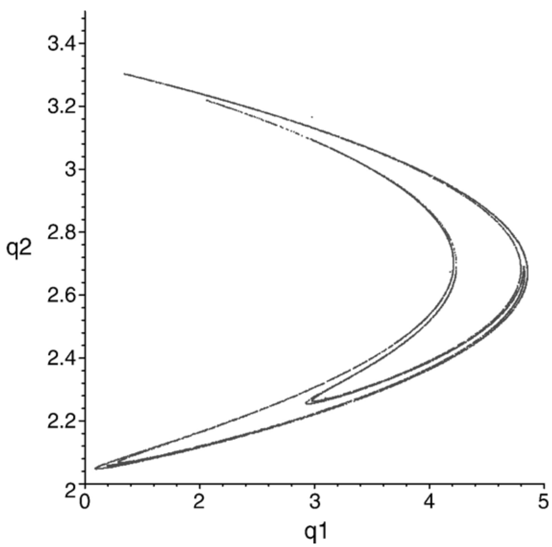

Fig. 2. The strange attractor of the discrete dynamic system (9).

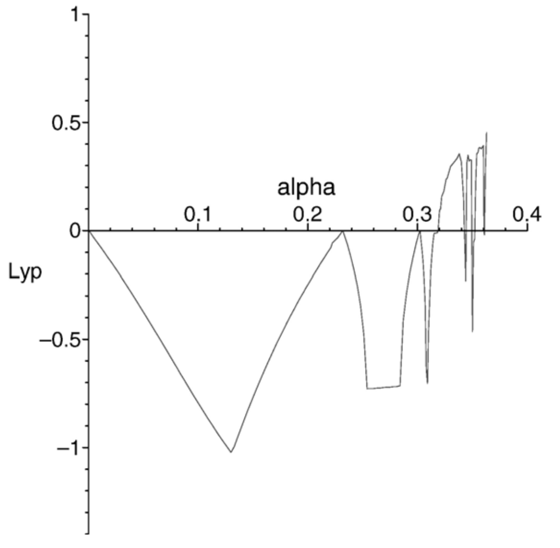

Fig. 3. Related the largest Lyapunov exponents as function of α.

doubling bifurcations of the quantity behavior become chaotic. It is clear that the period-doubling bifurcation occur at $\alpha=0.23$. When $\alpha>0.23$, we observe that flip bifurcation occurs and complex dynamic behavior begins to appear.

Fig. 2 shows the strange attractor for the system (9) for the values of $a=10$, $b=1$, $c_1=0.3$, $c_2=0.5$ and $\alpha=0.35$, which exhibits a fractal structure similar to Henon (1976) attractor.

In order to analyze the parameter sets for which aperiodic behavior occurs, we study the largest Lyapunov exponent, which depends on α. It is an evidence for chaos that the largest Lyapunov exponent is positive. By the method of von Bremen et al. (1997), we have Fig. 3 that displays the related maximal Lyapunov exponent as a function of α. From Fig. 3, we can easily get the degree of the local stability for different values α when the largest Lyapunov exponent is positive. We also determine the parameter sets for which the system (9) converges to cycles, aperiodic and chaotic behavior.

The sensitivity to initial conditions is a characteristic of chaos. In order to demonstrate the sensitivity to initial conditions of system (9), we compute two orbits with initial points (q_{10}, q_{20}) and $(q_{10}+0.00001, q_{20})$ at the parameter values $(a, b, c_1, c_2, \alpha)=(10, 1, 0.3, 0.5, 0.36)$, respectively. The results are plotted in Fig. 4.

The same to variable q_2, Fig. 5 shows the sensitivity dependence on initial conditions, q_2-coordinates of the two orbits with the parameter values $(a, b, c_1, c_2, \alpha)=(10, 1, 0.3, 0.5, 0.36)$; the q_2-coordinates of initial conditions differ by 0.00001.

Strange attractors are typically characterized by fractal dimensions. We examine the important characteristic of neighboring chaotic orbits to see how rapidly they separate each other. The Lyapunov dimension (Kaplan and Yorke, 1979) is defined as follows:

$$d_L = j + \frac{\sum_{i=1}^{i=j} \lambda_i}{|\lambda_j|}$$

with $\lambda_1, \lambda_2, \ldots, \lambda_n$, where j is the largest integer such that $\sum_{i=1}^{i=j} \lambda_i \geq 0$ and $\sum_{i=1}^{i=j+1} \lambda_i < 0$.

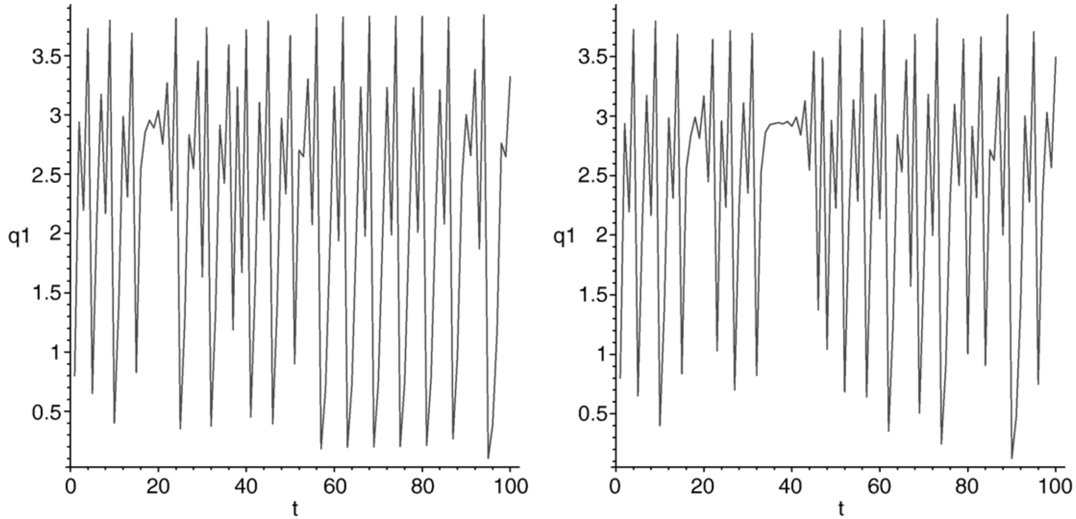

Fig. 4. Shows sensitive dependence on initial conditions, the two orbits of q_1-coordinates for $(a, b, c_1, c_2, \alpha)=(10, 1, 0.3, 0.5, 0.36)$.(1) $(q_{10}, q_{20})=(0.8, 0.5)$,(2)$(q_{10}, q_{20})=(0.80001, 0.5)$.

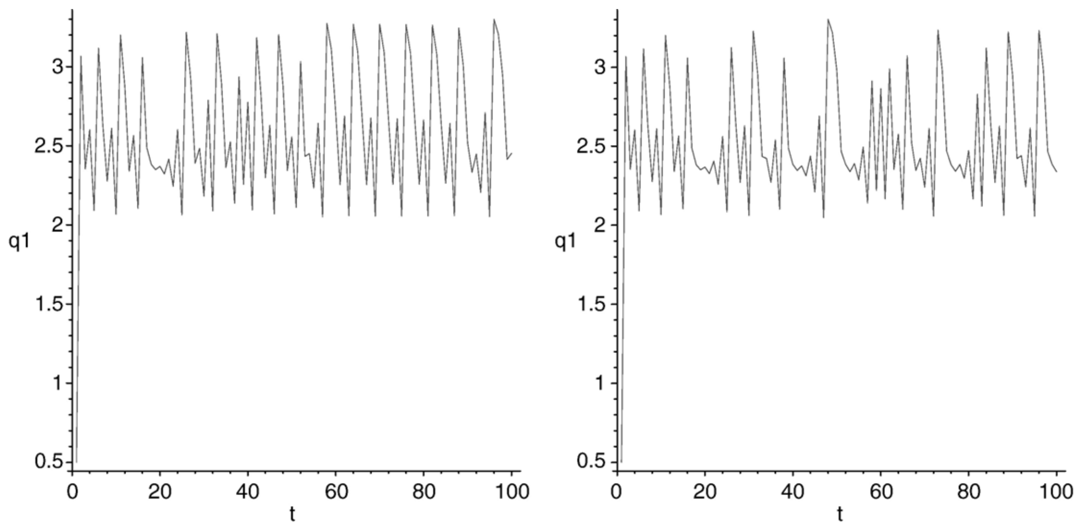

Fig. 5. Shows sensitive dependence on initial conditions, the two orbits of q_2-coordinates for $(a, b, c_1, c_2, \alpha)=(10, 1, 0.3, 0.5, 0.36)$. (1) $(q_{10}, q_{20})=(0.8, 0.5)$, (2) $(q_{10}, q_{20})=(0.8, 0.50001)$.

In our paper, the two-dimensional map (9) has a Lyapunov dimension

$$d_L = 1 + \frac{\lambda_1}{|\lambda_2|}, \lambda_1 > 0 > \lambda_2$$

By the definition of Lyapunov dimension (Kaplan and Yorke, 1979) and simulation of the computer, we have the Lyapunov dimension of the strange attractor of system (9). At the parameter values $(a, b, c_1, c_2, \alpha)=(10, 1, 0.3, 0.5, 0.36)$, system (9) has two different Lyapunov exponents, $\lambda_1=0.39$ and $\lambda_2=-1.92$. Therefore, the system (9) has a fractal dimension $d_L=1+(0.39/1.92)\approx 1.2$. Then the system (9) exhibits a fractal structure and its attractor has the fractal dimension $d_L\approx 1.2$.

5. Conclusion

In this paper, we analyzed the complex dynamics of a nonlinear, duopoly game with nonlinear cost function, which contains two kinds of heterogeneous players: boundedly rational player and naive player. The stability of equilibria, bifurcation and chaotic behavior are investigated in this game. We show that the speed of adjustment of boundedly rational player may change the stability of equilibria and cause a market structure to behave chaotically. For the low value speeds of adjustment, the game has a stable Nash equilibrium. Increasing the values of speeds of adjustment, the Nash equilibrium becomes unstable, through period-doubling bifurcation, more complex attractors obtained, which may be periodic cycles or chaotic sets.

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