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Physica A 320 (2003) 512–524
PHYSICA A
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Nonlinear dynamics in the Cournot duopoly game with heterogeneous players

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Received 31 May 2002

Abstract

We analyze a nonlinear discrete-time Cournot duopoly game, where players have heterogeneous expectations. Two types of players are considered: boundedly rational and naive expectations. In this study we show that the dynamics of the duopoly game with players whose beliefs are heterogeneous, may become complicated. The model gives more complex chaotic and unpredictable trajectories as a consequence of increasing the speed of adjustment of boundedly rational player. The equilibrium points and local stability of the duopoly game are investigated. As some parameters of the model are varied, the stability of the Nash equilibrium point is lost and the complex (periodic or chaotic) behavior occurs. Numerical simulations are presented to show that players with heterogeneous beliefs make the duopoly game behave chaotically. Also, we get the fractal dimension of the chaotic attractor of our map which is equivalent to the dimension of Henon map.

PACS: 87.23.Ge; 05.45.Gg; 05.45.Pq

Keywords: Discrete dynamical systems; Cournot duopoly games; Complex dynamics; Heterogeneous expectations

1. Introduction

An oligopoly is a market system which is controlled by a few number of firms producing homogeneous products. The dynamic of oligopoly game is more complex because oligopolist must consider not only the behaviors of the consumers, but also

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PII: S0378-4371(02)01648-5

the reactions of the other competitors. Cournot, in 1838 [1], introduced the first formal theory of oligopoly, who treated the case with naive expectations, so that in every step each player assumes the last values taken by the competitors without estimation of their future reactions.

Recently, several works have shown that the Cournot model may lead to complex behaviors such as cyclic and chaotic, see, for example Refs. [2–7]. Among the first to do this was Puu [3,4] who found a variety of complex dynamics arising in the Cournot duopoly case including the appearance of attractors with fractal dimension. Other studies on the dynamics of oligopoly models with more firms and other modifications include Ahmed and Agiza [8], Agiza [5] and Agiza et al. [9] such efforts have been extended by Bischi and Kopel [10] in a duopoly game with adaptive expectations. The development of complex oligopoly dynamics theory have been reviewed in Ref. [11].

Expectations play a key role in modelling economics phenomena. A producer can choose his expectations rules of many available techniques to adjust his production outputs. May be in the market of duopoly model each firm behaves with different expectations strategies, so we are going to apply this kind of expectations in our model which is common in reality.

In this paper we consider a duopoly model which is introduced in Ref. [7] but each player form a different strategy in order to compute his expected output. We take firm 1 represent a boundedly rational player while firm 2 has naive expectations. Each player adjusts his outputs towards the profit maximizing amount as target and use his expectations rule. Recently, examples of oligopoly games with homogeneous expectations are studied by Puu [4], Kopel [6], Agiza [12], Agiza et al. [13,14]. It was shown that the dynamics of Cournot oligopoly game may never settle to a steady state, and in the long run they exhibit bounded dynamic which may be periodic or chaotic. Economic model with heterogeneous players is introduced see [15,16]. Also, the dynamics of heterogeneous two-dimensional cobweb model have been studied by Onozaki et al., see Ref. [17].

The main aim of this work is to investigate the dynamic behaviors of a heterogeneous model representing two firms using heterogeneous expectations rules. This mechanism was applied in cobweb model [17] and gave us a guide to apply it in our study.

The paper is organized as follows. In Section 2 we describe the evolution of dynamical systems of players with heterogeneous expectations rules. In Section 3, the dynamics of a duopoly game with boundedly rational player and naive player is modelled by a two-dimensional map. The existence and local stability of the equilibrium points of the nonlinear map are analyzed. Complex dynamics of behavior occur under some changes of control parameters of the model which are shown by numerical experiments. Fractal dimension of the strange attractor of the map is measured numerically.

2. Heterogeneous expectations

In oligopoly game players choose simple expectations such as naive or complex as rational expectations. The players can use the same strategy (homogeneous expectations) or use different strategies (heterogeneous expectations). Several economic models represent the dynamics of heterogeneous firms, have been proposed in recent years see, for example, Refs. [17,18]. In this study we consider duopoly game where each player has different strategy to maximize his profit.

Firms can use rational expectations if they assume perfect knowledge of underlying market and this may not be available in real economic market. Also, it is well known that in a duopoly model with a heterogeneous firms their outputs depend upon expectations of all competitors. Hence, rational expectations can only be achieved under unrealistic assumptions. For this reason firms try to use another and my more realistic method which is called bounded rationality. Firms usually do not have a complete knowledge of the market, hence they try to use partial information based on the local estimates of the marginal profit. At each time period t each firm increases (decreases) its production q_i at the period $(t+1)$ if the marginal profit is positive (negative). If the players use this kind of adjustments then they are boundedly rational players.

Let us consider a Cournot duopoly game where q_i denotes the quantity supplied by firm $i, i=1,2$. In addition let $P(q_i+q_j)$, $i \neq j$, denote a twice differentiable and nonincreasing inverse demand function and let $C_i(q_i)$ denote the twice differentiable increasing cost function. Hence the profit of firm i is given by

$$\Pi_i = P(q_i+q_j)q_i - C_i(q_i)\ . \tag{1}$$

At each time period every player must form an expectation of the rival's output in the next time period in order to determine the corresponding profit-maximizing quantities for period $t+1$. If we denote by $q_i(t)$ the output of firm i at time period t, then its production $q_i(t+1)$, $i=1,2$ for the period $t+1$ is decided by solving the two optimization problems.

$$\begin{aligned} q_1(t+1) &= \arg\max_{q_1} \Pi_1(q_1(t), q_2^e(t+1))\ , \\ q_2(t+1) &= \arg\max_{q_2} \Pi_2(q_1^e(t+1), q_2(t))\ , \end{aligned} \tag{2}$$

where the function $\Pi_i(.,.)$ denotes the profit of the i the firm and $q_j^e(t+1)$ represents the expectation of firm i about the production decision of firm j, $(j=1,2, j\neq i)$. Cournot [1] assumed that $q_i^e(t+1) = q_i(t)$, firm i expects that the production of firm j will remain the same as in current period (naive expectations). The solution of the optimization problem of producer i can be expressed as $q_1(t+1) = f(q_2(t))$, and $q_2(t+1) = g(q_1(t))$, so that the time evolution of the duopoly system is obtained by the iteration of the two-dimensional map $T: R^2 \to R^2$ given by

$$T: \begin{cases} q_1' = f(q_2)\ , \\ q_2' = g(q_1)\ . \end{cases} \tag{3}$$

where $'$ represents the one-period advancement operator.

The map (3) describes the duopoly game in the case of homogeneous expectations (naive). The fixed points of the map (3) are located at the intersections of the two reaction functions $q_1 = f(q_2)$ and $q_2 = f(q_1)$ and are called Cournot–Nash equilibria of the two-players game.

Firms try to use more complex expectations such as bounded rationality [19], hence they try to use local information based on the marginal profit $\partial\Pi_i/\partial q_i$. At each time period t each firm increases (decreases) its production q_i at the period $(t+1)$ if the marginal profit is positive (negative). If the players use this kind of adjustments then they are boundedly rational players and the dynamical equation of this game has the form

$$q_i(t+1) = q_i(t) + \alpha_i q_i(t)\frac{\partial \Pi_i}{\partial q_i(t)}, \quad t = 0, 1, 2, \dots, \tag{4}$$

where α_i is a positive parameter which represents the speed of adjustment. The dynamics of the game (4) was studied by Bischi and Naimzada [7].

In duopoly game with players use a different expectations, for example the first player is boundedly rational player and the other is naive. Hence the duopoly game in this case are composed from the first equation of (4) and the second equation of (3). Thus, the discrete dynamical system in this case is described by

$$\begin{aligned} q_1(t+1) &= q_1(t) + \alpha_1 q_1(t)\frac{\partial \Pi_1}{\partial q_1(t)}, \\ q_2(t+1) &= g(q_1(t)). \end{aligned} \tag{5}$$

Therefore, Eq. (5) describes the dynamics of a duopoly game with two players using heterogeneous expectations. In the next section we are going to apply this technique to a duopoly model with linear demand and cost functions.

3. Model

Let $q_i(t),\ i = 1, 2$ represents the output of ith supplier during period t, with a production cost function $C_i(q_i)$. The price prevailing in period t is determined by the total supply $Q(t) = q_1(t) + q_2(t)$ through a linear demand function

$$P = f(Q) = a - bQ, \tag{6}$$

where a and b positive constants of demand function. The cost function is taken in the linear form

$$C_i(q_i) = c_i q_i, \quad i = 1, 2, \tag{7}$$

where c_i is the marginal cost of ith firm. With these assumptions the single profit of ith firm is given by

$$\Pi_i = q_i(a - bQ) - c_i q_i, \quad i = 1, 2. \tag{8}$$

Then the marginal profit of ith firm at the point (q_1, q_2) of the strategy space is given by

$$\frac{\partial \Pi_i}{\partial q_i} = a - c_i - 2bq_i - bq_j, \quad i, j = 1, 2, \ j \neq i. \tag{9}$$

This optimization problem has unique solution in the form

$$q_i = r_i(q_j) = \frac{1}{2b}(a - c_i - bq_j). \tag{10}$$

If the two firms are naive players, then the duopoly game is describe from Eq. (3) by using Eq. (10) which has a linear form and the Nash equilibrium is asymptotically stable [6]. In this study we consider two players with different expectation, which the first is boundedly rationality player and the other naive player. The dynamic equation of the first player (boundedly rational player) is obtained from inserting (9) in (3) which has the form

$$q_1(t+1) = q_1(t) + \alpha q_1(t)(a - c_1 - 2bq_1(t) - bq_2(t)) . \tag{11}$$

Using second equation of Eq. (3) the second player (naive) updates his output according to the dynamic equation

$$q_2(t+1) = \frac{1}{2b}(a - c_2 - bq_1(t)) . \tag{12}$$

Then the duopoly game with heterogeneous players is described by a two-dimensional nonlinear map

$$T(q_1, q_2) \to (q_1', q_2')$$

which is defined from coupling the dynamic Eqs. (11) and (12) as follows:

$$T : \begin{cases} q_1' = q_1 + \alpha q_1(a - c_1 - 2bq_1 - bq_2) , \\ q_2' = \frac{1}{2b}(a - c_2 - bq_1) . \end{cases} \tag{13}$$

The map (13) is an invertable map of the plane. The study of the dynamical properties of the map (13) allows us to have information on the long-run behavior of heterogeneous players. Starting from given initial condition (q_{1_0}, q_{2_0}), the iteration of Eq. (13) uniquely determines a trajectory of the states of firms output. $(q_1(t), q_2(t)) = T^t(q_{1_0}, q_{2_0}),\ t = 0, 1, 2, \ldots$.

3.1. Equilibrium points and local stability

The fixed points of the map (13) are obtained as nonnegative solutions of the algebraic system

$$q_1(a - c_1 - 2bq_1 - bq_2) = 0 ,$$

$$(a - c_2 - 2bq_2 - bq_1) = 0$$

which is obtained by setting $q_i' = q_i,\ i = 1, 2$ in Eq. (13). We can have at most two fixed points $E_0 = (0, (a - c_2)/2b)$ and $E_* = (q_1^*, q_2^*)$. The fixed point E_0 is called a boundary equilibrium [7] and have economic meaning when $c_2 < a$. The second equilibrium E_* is called Nash equilibrium where

$$q_1^* = \frac{a + c_2 - 2c_1}{3b} \quad \text{and} \quad q_2^* = \frac{a + c_1 - 2c_2}{3b} \tag{14}$$

provided that

$$2c_1 - c_2 < a ,$$

$$2c_2 - c_1 < a . \tag{15}$$

It is easy to verify that the equilibrium point E_* is located at the intersection of the two reaction curves which represent the locus of points of vanishing marginal profit in Eq. (9). In the following, we assume that Eq. (15) is satisfied, so the Nash equilibrium E_* exists.

The study of the local stability of equilibrium solutions is based on the localization, on the complex plane of the eigenvalues of the Jacobian matrix of the two-dimensional map (Eq. (13)).

The study of the local stability of equilibrium solutions is based on the localization, on the complex plane of the eigenvalues of the Jacobian matrix of the two-dimensional map (Eq. (13)).

The Jacobian matrix of the map (13) at the state (q_1, q_2) has the from

$$J(q_1,q_2) = \begin{bmatrix} 1+\alpha(a-4bq_1-bq_2-c_1) & -\alpha bq_1 \\ \frac{-1}{2} & 0 \end{bmatrix} . \tag{16}$$

The determinant of the matrix J is

$$Det = -\tfrac{1}{2}\,\alpha bq_1 \ .$$

Hence the map (13) is dissipative dynamical system when $|\alpha bq_1| < 2$.

Lemma 1. *The fixed point E_0 of the map (Eq. (13)) is unstable.*

Proof. In order to prove this results, we estimate the eigenvalues of Jacobian matrix J at E_0. The Jacobian matrix has the form

$$J(E_0) = \begin{bmatrix} 1+\dfrac{\alpha}{2}(a-2c_1+c_2) & 0 \\ -\frac{1}{2} & 0 \end{bmatrix} .$$

The matrix $J(E_0)$ has two eigenvalues $\lambda_1 = 1 + (\alpha/2)(a - 2c_1 + c_2)$ and $\lambda_2 = 0$. From condition (15), it follows that $|\lambda_1| > 1$. Then E_0 is unstable fixed point (saddle point) for the map (13) and this completes the proof. □

3.1.1. Local stability of Nash equilibrium

We study the local stability of Nash equilibrium of two-dimensional map (13). The Jacobian matrix (16) at E_*, which take the form

$$J(E_*) = \begin{bmatrix} 1-2\alpha bq_1^* & -\alpha bq_1^* \\ -\frac{1}{2} & 0 \end{bmatrix} . \tag{17}$$

The characteristic equations of $J(E_*)$ is $P(\lambda) = \lambda^2 - Tr\,\lambda + Det = 0$ where Tr is the trace and Det is the determinant of the Jacobian matrix defines in (17), $Tr = 1 - 2\alpha bq_1^*$ and $Det = -\frac{1}{2}\alpha bq_1^*$.

Since $(Tr)^2 - 4Det = (1 - 2\alpha bq_1^*)^2 + 2\alpha bq_1^*$. It is clear that $(Tr)^2 - 4Det > 0$ (has positive discriminant), then we deduce that the eigenvalues of Nash equilibrium are real. The local stability of Nash equilibrium is given by using Jury's conditions [21] which are:

1. $|Det| < 1$
2. $1 - Tr + Det > 0$, and
3. $1 + Tr + Det > 0$.

The first condition is $|\alpha b q_1^*| < 2$, which implies that

$$\alpha < \frac{6}{(a - 2c_1 + c_2)} . \tag{18}$$

The second condition $1 - Tr + Det = \frac{3}{2}\alpha b q_1^* > 0$, then the second condition is satisfied. Then the third condition becomes

$$\tfrac{5}{2}\alpha b q_1^* - 2 < 0 .$$

This inequality is equivalent to

$$\alpha < \frac{12}{5(a - 2c_1 + c_2)} . \tag{19}$$

Form (18) and (19), it follows that the Nash equilibrium is stable if $\alpha < 12/5(a - 2c_1 + c_2)$ and hence the following lemma is proved. □

Lemma 2. *The Nash equilibrium E_* of the map (Eq. (13)) is stable provided that $\alpha < 12/5(a - 2c_1 + c_2)$.*

From the previous lemma, we obtain information of the effects of the model parameters on the local stability of Nash equilibrium point E_*. For example, an increase of the speed of adjustment of boundedly rational player with the other parameters held fixed, has a destabilizing effect. In fact, an increase of α, starting from a set of parameters which ensures the local stability of the Nash equilibrium, can bring out the region of the stability of Nash equilibrium point, crossing the flip bifurcation surface $\alpha = 12/5(a - 2c_1 + c_2)$. Similar argument apply if the parameters α, b, c_1 and c_2 are fixed parameters and the parameter a, which represents the maximum price of the good produced, is increased. In this case the region of stability of E_* becomes small, and this implies that E_* losses its stability. Complex behaviors such as period doubling and chaotic attractors are generated where the maximum Lyapunov exponents of the map (13) become positives.

3.2. Numerical investigations

The main purpose of this section is to show that the qualitative behavior of the solutions of the duopoly game (13) with heterogeneous player generates a complex behavior that the case of duopoly game with homogeneous (naive) player.

To provide some numerical evidence for the chaotic behavior of system (13), we present various numerical results here to show the chaoticity, including its bifurcations diagrams, strange attractors, Laypunov exponents, Sensitive dependence on initial conditions and fractal structure. In order to study the local stability properties

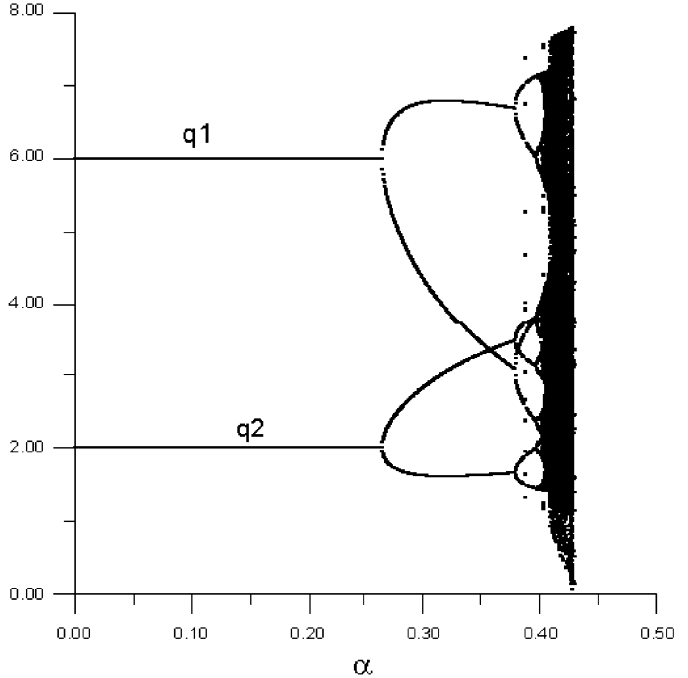

Fig. 1. Bifurcation diagram with respect to the parameter α speed of adjustment of bounded rational player, with other fixed parameters $a=10, b=0.5, c_1=3$, and $c_2=5$.

of the equilibrium points, it is convenient to take the parameters values as follows: $a=10, b=0.5, c_1=3$ and $c_2=5$.

Fig. 1 shows the bifurcation diagram with respect to the parameter α (speed of adjustment of boundedly rational player), while the other parameters are fixed ($a=10, b=0.5, c_1=3$, and $c_2=5$). In fact a bifurcation diagram of a two-dimensional map (13) shows attractor of the model (13) as a multi-valued of two-dimensional map of one parameter. In Fig. 1, the bifurcation scenario is occurred, if α is small then three exists a stable equilibrium point (Nash). As one can see the Nash equilibrium point $(6,2)$ is locally stable for small values of α. As α increases, the Nash equilibrium becomes unstable, infinitely many period doubling bifurcations of the quantity behavior becomes chaotic, as α increased. It means for a large values of speed of adjustment of bounded rational player α, the system converge always to complex dynamics. Also, one can see that the period doubling bifurcation occur at $\alpha=\frac{4}{15}$. If the case of $\alpha>\frac{4}{15}$, one observes flip bifurcation occurs and complex dynamic behavior begin to appear for $\frac{4}{15}<\alpha<1$.

A bifurcation diagram with respect to the marginal cost of the first player c_1, while other parameters are fixed as follows $a=10, b=0.5$, $c_2=5$ and $\alpha=0.335$, is shown in Fig. 2.

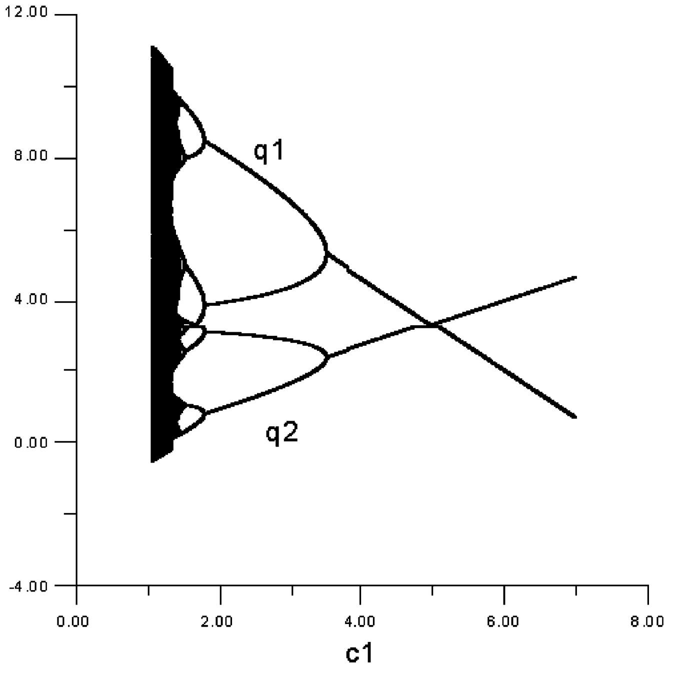

Fig. 2. A bifurcation diagram with respect to the marginal cost of the first player c_1, with other parameters fixed at $a = 10, b = 0.5,\ c_2 = 5$ and $\alpha = 0.335$.

We show the graph of a strange attractor for the parameter constellation $(a, b, c_1, c_2, \alpha) = (10, 0.5, 3, 5, 0.42)$ in Fig. 3, which exhibits a fractal structure similar to Henon attractor [22].

In order to analyze the parameter sets for which aperiodic behavior occurs, one can compute the maximal Lyapunov exponent depend on α. For example, if the maximal Lyapunov exponent is positive, one has evidence for chaos. Moreover, by comparing the standard bifurcation diagram in α, one obtains a better understanding of the particular properties of the system. In order to study the relations between the local stability of the Nash equilibrium point and the speed of adjustment of boundedly rational player α, one can compute the maximal Lyapunov exponents for adjustment factor in the environment of 1. Fig. 4 displays the related maximal Lyapunov exponents as a function of α. From Fig. 4, one can easily determine the degree of the local stability for different values of $\alpha \in (\frac{4}{15}, 1)$. At value of $\alpha > \frac{4}{15}$ the maximal Lyapunov exponents is positive. A positive value of maximal Lyapunov exponents implies sensitive dependence on initial condition for chaotic behavior. From the maximal Lyapunov exponents diagram it is easy to determine the parameter sets for which the system converges to cycles, aperiodic, chaotic behavior. Beyond that its even possible to differentiate between cycles of very higher order and aperiodic behavior of the map (13) see Fig. 4.

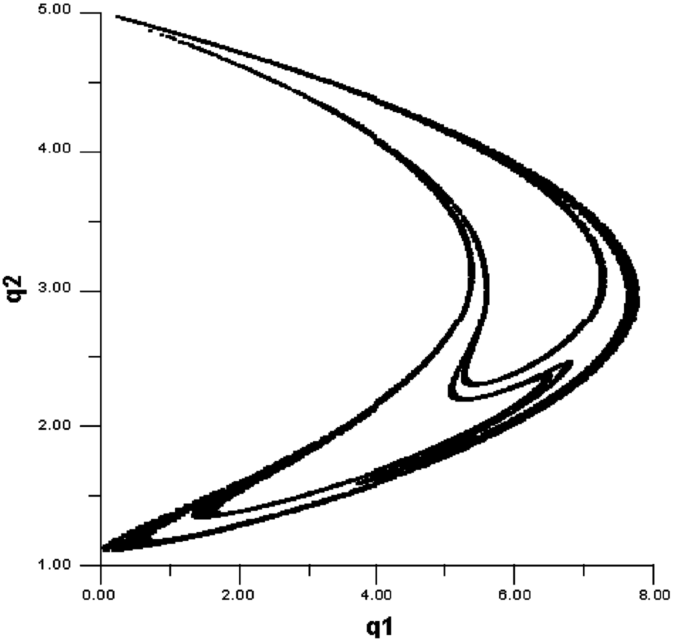

Fig. 3. We show the graph of a strange attractor for the parameter constellation $(a, b, c_1, c_2, \alpha) = (10, 0.5, 3, 5, 0.41)$.

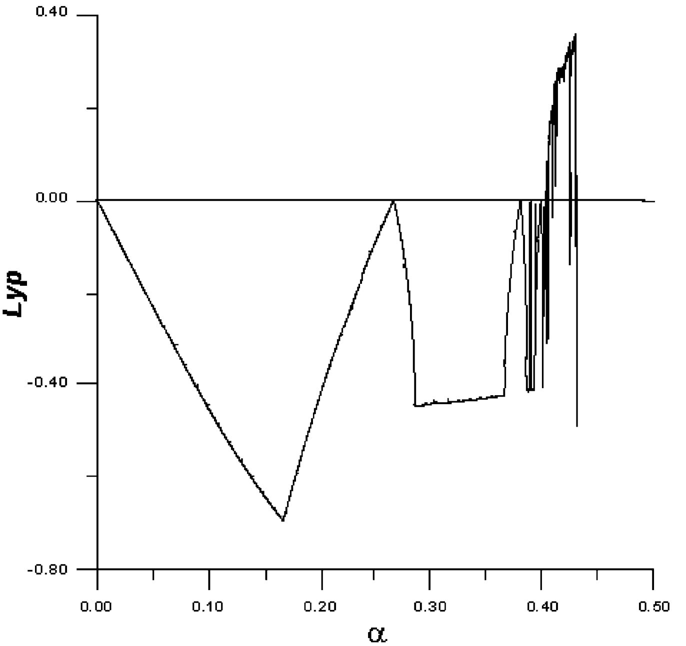

Fig. 4. Related maximal Lyapunov exponents as a function of α.

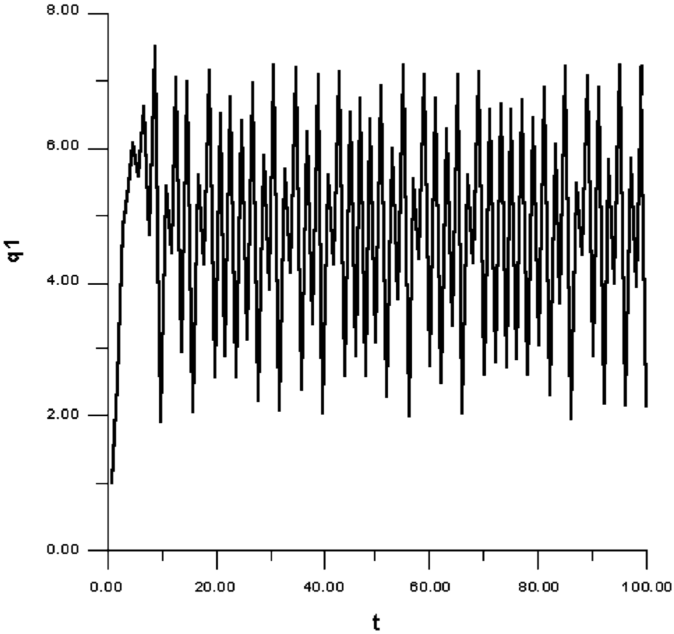

Fig. 5. For sensitive dependence on initial conditions: for system (13), parameter values $(a, b, c_1, c_2, \alpha) = (10, 0.5, 3, 5, 0.4)$. and initial conditions (q_{10}, q_{20}).

3.2.1. Sensitive dependence on initial conditions

To demonstrate the sensitivity to initial conditions of system (13), we compute two orbits with initial points (q_{1_0}, q_{2_0}) and $(q_{1_0} + 0.0001, q_{2_0})$, respectively. The results are shown in Figs. 5 and 6. At the beginning the time series are indistinguishable; but after a number of iterations, the difference between them builds up rapidly.

Figs. 5 and 6 show sensitive dependence on initial conditions, q_1-coordinates of the two orbits, for system (13), plotted against the time with the parameter constellation $(a, b, c_1, c_2, \alpha) = (10, 0.5, 3, 5, 0.4)$; the q_1-coordinates of initial conditions differ by 0.0001, the other coordinate kept equal.

3.2.2. Fractal dimension of the map (13)

Strange attractors are typically characterized by fractal dimensions. We examine the important characteristic of neighboring chaotic orbits to see how rapidly they separate each other. The Lyapunov dimension see Refs. [20,23] is defined as follows:

$$d_L = j + \frac{\sum_{i=1}^{i=j} \lambda_i}{|\lambda_j|}$$

with $\lambda_1, \lambda_2, \ldots, \lambda_n$, where j is the largest integer such that $\sum_{i=1}^{i=j} \lambda_i \geqslant 0$ and $\sum_{i=1}^{i=j+1} \times \lambda_i < 0$.

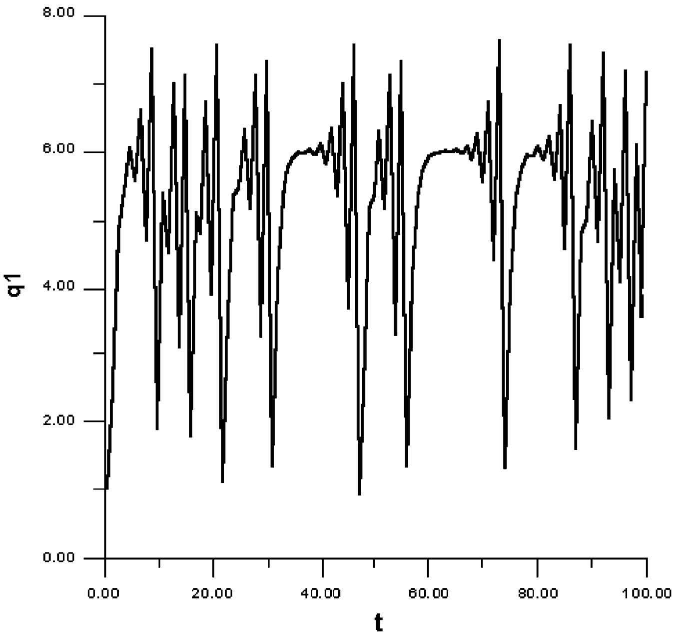

Fig. 6. For sensitive dependence on initial conditions for system (13), parameter values $(a, b, c_1, c_2, \alpha) = (10, 0.5, 3, 5, 0.4)$ and initial conditions $(q_{10} + 0.0001, q_{20})$.

In our case of the two-dimensional map has the Lyapunov dimension which is given by

$$d_L = 1 + \frac{\lambda_1}{|\lambda_2|}, \quad \lambda_1 > 0 > \lambda_2 \; .$$

By the definition of the Lyapunov dimension and with help of the computer simulation one can show that the Lyapunov dimension of the strange attractor of system (13). At the parameters values $(a, b, c_1, c_2, \alpha) = (10, 0.5, 3, 5, 0.41)$ two Lyapunov exponents exists and are $\lambda_1 = 0.28$ and $\lambda_2 = -1.06$. Therefore, the map (13) has a fractal dimension $d_L \approx 1 + \frac{0.28}{1.06} \approx 1.26$, which is the same fractal dimension of Henon map [24].

4. Conclusions

We have investigated the dynamics of a nonlinear, two-dimensional duopoly game, which contains two-types of heterogeneous players: boundedly rational player and naive player. This game is described by a two-dimensional invertible map. The stability of equilibria, bifurcation and chaotic behavior are analyzed. The influence of the main parameters (such as the speed of adjustment of boundedly rational player, the maximum price of demand function and the marginal costs of players) on the local stability is studied. We deduced that introducing heterogeneous expectations for players in the duopoly game cause a market structure to behave chaotically.

Acknowledgements

We thank T. Onazaki for his comments.

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